MATH 5010: Linear Analysis: Test

Answer ALL Questions:

5 points for each part 04 Nov 2017. 9:30-11:00

1. Suppose that the Euclidean space $\mathbb{R}^n$ is endowed with the usual norm, that is, $\|x\|_2 := \sqrt{\sum_{k=1}^n |x_k|^2}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

   (i) For each $x \in \mathbb{R}^n$, put $\|x\|_\infty := \max\limits_{1\leq k\leq n} |x_k|$. Using the definition of equivalent norms, show that the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent on $\mathbb{R}^n$.
   Proof: Notice that we always have $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$ for all $x \in \mathbb{R}^n$ and thus, $\|\cdot\|_\infty \sim \|\cdot\|_2$. □

   (ii) Let $z = (z_1, ..., z_n) \in \mathbb{R}^n$. Define an element $f \in (\mathbb{R}^n)^*$ by $f(x) = \sum_{k=1}^n x_k z_k$ for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Find $\|f\|$.
   Proof: Recall that we always have the Cauchy-Schwarz inequality: $|\sum_{k=1}^n x_k z_k| \leq \|x\|_2\|z\|_2$. This implies that $\|f\| \leq \|z\|_2$. On the other hand, it is clear that $f = 0$ as $z = 0$. Now suppose that $z \neq 0$. Then we have $f(z/\|z\|) = \sum_{k=1}^n \frac{z_k}{\|z\|} z_k = \|z\|$. So, we have $\|f\| = \|z\|_2$. □

2. Let $(X, \|\cdot\|)$ be a normed space. Suppose that $X = E \oplus F$ for some closed subspaces E and F of X, that is, $E \cap F = \{0\}$ and for each $x \in X$, there are elements $u \in E$ and $v \in F$ such that $x = u + v$. Now for each $x \in X$, write $x = u + v$ for some $u \in E$ and $v \in F$, define $\|x\|_1 := \|u\| + \|v\|$.

   (i) Show that $\|\cdot\|_1$ is a norm on X and F is also a closed subspace of X with respect to the norm $\|\cdot\|_1$.
   Proof: It can be directly shown by the definition. □

   (ii) Show that if $(X, \|\cdot\|)$ is a Banach space, then so is the normed space $(X, \|\cdot\|_1)$.
   Proof: Let (x_n) be a Cauchy sequence in X with respect to the norm $\|\cdot\|_1$. Put $x_n = e_n + f_n$ for $e_n \in E$ and $f_n \in F$. It is easy to see that the sequences (e_n) and (f_n) both are Cauchy sequences in E and F respectively in the norm $\|\cdot\|$. Since $(X, \|\cdot\|)$ is a Banach space and E, F both are closed, the limits $e := \lim e_n \in E$ and $f := \lim f_n \in F$ both exist with respect to the norm $\|\cdot\|$. If we let $x := e + f$, then by the definition of the norm $\|\cdot\|_1$, we have $\|x_n - x\|_1 \to 0$ as desired. □

   (iii) Write X_1 for the normed space $(X, \|\cdot\|_1)$. Define a linear map

   $$T : x \in E \mapsto \bar{x} \in X_1/F$$

   where $\bar{x}$ denotes the equivalence class of x in X_1/F. Show that T is bounded.
   Proof: We first claim that F is closed with respect to the norm $\|\cdot\|_1$. In fact, if (f_n) is a sequence in F such that the limit $u := \lim f_n$ exists with respect to the norm $\|\cdot\|_1$.

Write $u = e + f$ for $e \in E$ and $f \in F$. Then we have

$$\|e\| + \|f - f_n\| = \|u - f_n\|_1 \to 0.$$

So, $e = 0$ and thus, $u = f \in F$, that is, F is closed with respect to the norm $\|\cdot\|_1$. Therefore, the quotient space X_1/F is defined.

Now for $x \in E$, we have $\bar{x} := \inf\{\|x+y\|_1 : y \in F\}$. Therefore, we have $\|\bar{x}\| \leq \|x\|_1 = \|x\|$ and hence, $\|T\| \leq 1$. □

(iv) Show that the inverse $T^{-1} : X_1/F \to E$ of T in (iii) is also bounded.

(Hint: the inverse T^{-1} is given by

$$T^{-1}(\bar{x}) = u \quad \text{if } x = u + v \text{ for } u \in E \text{ and } v \in F. \tag{0.1}$$

Notice that you have to explain why the Eq. (0.1) is well defined.)

Proof: One can directly check that T is a linear isomorphism with the inverse T^{-1} given by Eq. (0.1).

Now let $\bar{x} \in X_1/F$ and $x = u+v$ for some $u \in E$ and $v \in F$. Note that if $y \in F$, then we have $\|x + y\|_1 = \|u\| + \|v + y\| \geq \|u\|$. Therefore, we have $\|u\| \leq \|\bar{x}\| := \inf\{\|x + y\|_1 : y \in F\}$. Thus, $\|T^{-1}\| \leq 1$.

(**Note:** in fact, from the proof above, we have shown that T is an isometric isomorphism from E onto X_1/F). □

End